Mini-courses in Mathematical Analysis 2021:

Spectral theory for Selfadjoint Partial Differential Operators

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June 17, 2021

Chapter 0

Resources, prerequisites and notation

Online resources

Expanded lecture notes covering a lot more topics
https://wiki.cites.illinois.edu/wiki/display/MATH595STP/Math+595+STP

Java applets for vibrating membranes (and more)
http://www.falstad.com/mathphysics.html

Prerequisites

We assume familiarity with elementary Hilbert space theory: inner product, norm, Cauchy–Schwarz, orthonormal basis (ONB), bounded and compact operators.

Function spaces

All functions are assumed measurable. Readers unfamiliar with Sobolev spaces need only know that:

$$\begin{aligned} \mathsf{L}^2 &= \{\text{square integrable functions}\}, \\ \mathsf{H}^1 = W^{1,2} &= \{\mathsf{L}^2\text{-functions with } 1 \text{ derivative in } \mathsf{L}^2\}, \\ \mathsf{H}^1_0 = W^{1,2}_0 &= \{\mathsf{H}^1\text{-functions that equal zero on the boundary}\}. \end{aligned}$$

These characterizations are not mathematically precise, but suffice for our purposes. Later we will recall the standard inner products that make these Sobolev spaces into Hilbert spaces. For more on Sobolev space theory, and related concepts of weak solutions and elliptic regularity, see [Evans].

Laplacian

$$\Delta = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_d}\right)^2$$

Divergence theorem Given a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth enough boundary, and a vector field F on the closure of the domain, one has

$$\int_\Omega \nabla \cdot F \, dx = \int_{\partial\Omega} F \cdot n \, dS$$

where n denotes the outward unit normal vector on $\partial\Omega$. In 1-dimension, the Divergence Theorem is simply the Fundamental Theorem of Calculus:

$$\int_{(a,b)} F'(x) \, dx = -F(a) + F(b)$$

where the negative sign indicates the leftward orientation of n at $x = a$.

Green's formulas

$$\int_\Omega (\nabla u \cdot \nabla v + v\Delta u) \, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, dS \tag{0.1}$$

$$\int_\Omega (u\Delta v - v\Delta u) \, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \tag{0.2}$$

where the normal derivative is defined as the normal component of the gradient vector:

$$\frac{\partial u}{\partial n} = \nabla u \cdot n.$$

Proof. Apply the Divergence theorem to $F = v\nabla u$. Interchange u and v, and subtract.

Integration by parts

$$\int_\Omega \frac{\partial u}{\partial x_j} v \, dx = -\int_\Omega u \frac{\partial v}{\partial x_j} \, dx + \int_{\partial\Omega} uv \, n_j \, dS$$

where n_j is the jth component of the normal vector.

Proof. Apply the Divergence Theorem to $F = (0, \ldots, 0, uv, 0, \ldots, 0)$.

Chapter 1

Motivation — ODE overture

Goal

To review the role of eigenvalues and eigenvectors in solving 1st and 2nd order systems of linear ODEs, to interpret eigenvalues as decay rates and frequencies, and to observe formal analogies with PDEs.

Notational convention

Eigenvalues are written with multiplicity, and are listed in increasing order (when real valued):

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

Spectrum of a real symmetric matrix

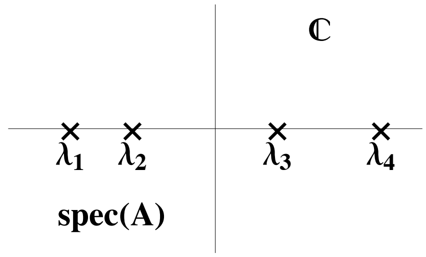
If A is a real symmetric $d \times d$ matrix (*e.g.* $A = \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ when $d = 2$) then its *spectrum* is the collection of eigenvalues:

$$\mathrm{spec}(A) = \{\lambda_1, \ldots, \lambda_d\} \subset \mathbb{R}$$

(see the figure). Recall that

$$Av_j = \lambda_j v_j$$

where the eigenvectors $\{v_1, \ldots, v_d\}$ can be chosen to form an ONB for $\mathbb{R}^d$.



Observe $A : \mathbb{R}^d \to \mathbb{R}^d$ is diagonal with respect to the eigenbasis:

$$A\Big(\sum c_j v_j\Big) = \sum \lambda_j c_j v_j$$

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix} = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_d c_d \end{bmatrix}$$

Corresponding statements hold for complex Hermitian matrices acting on $\mathbb{C}^d$.

What does the spectrum tell us about linear ODEs? Decay rates and frequencies. . .

Example 1.1 (1st order).

$$\begin{aligned} \frac{dv}{dt} &= -Av && \text{ODE} \\ v(0) &= \sum c_j v_j && \text{IC} \end{aligned}$$

has solution

$$v(t) = e^{-At} v(0) \stackrel{\text{def}}{=} \sum e^{-\lambda_j t} c_j v_j.$$

Notice

$$\lambda_j = \begin{cases} \textbf{decay rate} \text{ of the solution in direction } v_j, \text{ if } \lambda_j > 0, \\ \textbf{growth rate} \text{ of the solution in direction } v_j, \text{ if } \lambda_j < 0. \end{cases}$$

Long-time behavior: the solution is dominated by the first mode, with

$$v(t) \simeq e^{-\lambda_1 t} c_1 v_1 \qquad \text{for large } t$$

assuming $\lambda_1 < \lambda_2$ (so that the second mode decays faster than the first). The rate of collapse onto the first mode is governed by the **spectral gap** $\lambda_2 - \lambda_1$ since

$$\begin{aligned} v(t) &= e^{-\lambda_1 t}\Big(c_1 v_1 + \sum_{j=2}^{d} e^{-(\lambda_j - \lambda_1)t} c_j v_j\Big) \\ &= e^{-\lambda_1 t}\big(c_1 v_1 + O(e^{-(\lambda_2-\lambda_1)t}\big). \end{aligned}$$

Example 1.2 (2nd order)**.** Assume $\lambda_1 > 0$, so that all the eigenvalues are positive. Then the system

$$\begin{aligned} \frac{d^2 v}{dt^2} &= -Av && \text{ODE} \\ v(0) &= \sum c_j v_j && \text{IC displacement} \\ v'(0) &= \sum d_j v_j && \text{IC velocity} \end{aligned}$$

has solution

$$\begin{aligned} v(t) &= \cos(\sqrt{A}t)v(0) + \frac{1}{\sqrt{A}} \sin(\sqrt{A}t)v'(0) \\ &\stackrel{\text{def}}{=} \sum \cos(\sqrt{\lambda_j}t)c_j v_j + \sum \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t) d_j v_j. \end{aligned}$$

Notice $\sqrt{\lambda_j}$ = **frequency** of the solution in direction v_j.

Example 1.3 (1st order complex)**.** The system

$$\begin{aligned} i\frac{dv}{dt} &= Av && \text{ODE} \\ v(0) &= \sum c_j v_j && \text{IC} \end{aligned}$$

has complex valued solution

$$v(t) = e^{-iAt}v(0) \stackrel{\text{def}}{=} \sum e^{-i\lambda_j t} c_j v_j.$$

Here λ_j = **frequency** of the solution in direction v_j.

Looking ahead to PDEs

The negative Laplacian $A = -\Delta$ on a domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions is a linear operator that behaves in some ways like a self-adjoint matrix. As we will learn in Chapter 5, its eigenvalues λ_j and eigenfunctions $v_j(x)$ satisfy

$$\begin{aligned} -\Delta v_j &= \lambda_j v_j \qquad && \text{in } \Omega, \\ v_j &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and the spectrum is real and increases to infinity:

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty.$$

The eigenfunctions form an ONB for $L^2(\Omega)$.

Formally substituting $A = -\Delta$ into ODE Examples 1.1–1.3 transforms them into famous PDEs for the function $v(x,t)$, and transforms the formulas for v into "separation of variables" solutions:

- Example 1.1 — diffusion equation $v_t = \Delta v$, where $v(x,t)$ represents chemical concentration or temperature. Solution:

  $$v = e^{\Delta t} v(\cdot, 0) \overset{\text{def}}{=} \sum e^{-\lambda_j t} c_j v_j(x)$$

  where the initial value is $v(\cdot,0) = \sum c_j v_j$. Here λ_j = decay rate.

- Example 1.2 — wave equation $v_{tt} = \Delta v$, where $v(x,t)$ represents the vertical displacement at time t of a horizontal membrane, or the oscillation of an electromagnetic signal. Solution:

  $$\begin{aligned} v &= \cos(\sqrt{-\Delta}\, t) v(\cdot,0) + \frac{1}{\sqrt{-\Delta}} \sin(\sqrt{-\Delta}\, t) v_t(\cdot,0) \\ &\overset{\text{def}}{=} \sum \cos(\sqrt{\lambda_j}\, t) c_j v_j(x) + \sum \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}\, t) d_j v_j(x). \end{aligned}$$

  In this case $\sqrt{\lambda_j}$ = frequency and v_j = mode of vibration.

- Example 1.3 — Schrödinger equation $iv_t = -\Delta v$, where $|v(x,t)|^2$ represents the probability density at time t for the location of a quantum particle. Solution:

  $$v = e^{i\Delta t} v(\cdot,0) \overset{\text{def}}{=} \sum e^{-i\lambda_j t} c_j v_j(x).$$

  Here λ_j = frequency or energy level, and v_j = quantum state.

Remark. The methods that will be covered in this minicourse can handle not only the Laplacian, but a whole family of related operators including:

$$
\begin{array}{ll}
A = -\Delta & \text{Laplacian,} \\
A = -\Delta + V(x) & \text{Schrödinger operator,} \\
A = (i\nabla + \vec{V})^2 & \text{magnetic Laplacian,} \\
A = (-\Delta)^2 = \Delta\Delta & \text{biLaplacian.}
\end{array}
$$

The spectral theory of these operators helps to solve the corresponding evolution equations, and explain the stability or instability of different types of equilibrium solutions, namely: steady states, standing waves, traveling waves, and similarity solutions.

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